MATH4060 Tutorial 6

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Problem 1 (Chap 7, Ex 1). *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums $A_n = a_1 + \cdots + a_n$ are bounded. Prove that the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ *and defines a holomorphic function in this half-plane.*

To apply Theorem 5.2 of Chapter 2, we want to show that the series is uniformly convergent on any compact subset of the half-plane. Assume $|A_n| \leq M$ for all $n \in \mathbb{N}$. Using summation by parts, for $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \frac{A_N}{N^s} + \sum_{n=1}^{N-1} A_n(n^{-s} - (n+1)^{-s}).$$

Since $|A_N/N^s| \leq M/N^{\mathrm{Re}(s)} \to 0$ uniformly on any closed half-plane $\mathrm{Re}(s) \geq \delta > 0$ as $N \to \infty$, it suffices to show that the series $\sum A_n(n^{-s} - (n+1)^{-s})$ is uniformly convergent on any compact subset of $\mathrm{Re}(s) > 0$. Let $g(z) = z^{-s}$ so that $g'(z) = -sz^{-s-1}$. By considering $z(t) = n + t$, $t \in [0, 1]$, we have

$$|(n+1)^{-s} - n^{-s})| = \left| \int_0^1 g'(z(t))z'(t)\, dt \right| \leq |s| \int_0^1 (n+t)^{-\mathrm{Re}(s)-1}\, dt \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$$

On any compact set K, $|s| \leq B$ and $\mathrm{Re}(s) \geq \delta$ for some $B, \delta > 0$, so

$$\sum_{n=1}^{\infty} |A_n(n^{-s} - (n+1)^{-s})| \leq \sum_{n=1}^{\infty} \frac{M|s|}{n^{\mathrm{Re}(s)+1}} \leq MB \sum_{n=1}^{\infty} \frac{1}{n^{\delta+1}}$$

is uniformly convergent on K.

Problem 2 (Chap 7, Ex 5). *Consider the following function*

$$\tilde{\zeta}(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

(a) *Prove that the series defining $\tilde{\zeta}$ converges for* $\mathrm{Re}(s) > 0$ *and defines a holomorphic function in that half-plane.*

(b) *Show that for $s > 1$ one has $\tilde{\zeta}(s) = (1 - 2^{1-s})\zeta(s)$.*

(c) *Conclude, since $\tilde{\zeta}$ is given as an alternating series, that ζ has no zeros on the segment $0 < s < 1$. Extend this last assertion to $s = 0$ by using the functional equation.*

(a) Since partial sums of $\sum(-1)^{n+1}$ are certainly bounded, the previous problem applies.

(b) On $s > 1$, as $\zeta(s)$ and $\tilde{\zeta}(s)$ are absolutely convergent (as series), we compute that

$$\zeta(s) - \tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = 2^{1-s}\zeta(s).$$

(c) Notice that at $s = 1$, the simple pole of $\zeta(s)$ cancels with the zero of $1 - 2^{1-s}$, so both sides of the identity in (b) are holomorphic functions on $\mathrm{Re}(s) > 0$ that agree on $s > 1$. Thus the identity holds on the whole half-plane. Focusing on $0 < s < 1$, we have

$$\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s} > 0$$

for $n \in \mathbb{N}$, so $\tilde{\zeta}(s) > 0$, and hence $\zeta(s) \neq 0$ on $0 < s < 1$ by the identity. Finally, using the functional equation

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s),$$

we see that at $s = 0$, the simple pole of $\zeta(1-s)$ cancels with the simple zero of $1/\Gamma(s/2)$, so the RHS is nonzero. This concludes that $\zeta(s) \neq 0$ on $[0, 1)$.

Problem 3 (cf. Chap 7, Ex 8). *Show that ζ has infinitely many zeros in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$.*

We first prove that the entire function $\tilde{\xi} = s(1-s)\xi(s)$ has growth order 1. To show that $\rho_{\tilde{\xi}} \leq 1$, we shall use the representation

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{-s/2-1/2} + u^{s/2-1})\psi(u)\, du,$$

where $\psi(u) = \sum_{n=1}^\infty e^{-\pi n^2 u}$. Because $s(1-s)$ is a polynomial, it suffices to show that the integral term in $\xi(s)$ defines an entire function of growth ≤ 1. For $s = \sigma + it \in \mathbb{C}$, take any $k \in \mathbb{N}$ such that $(|\sigma| + 1)/2 \leq k \leq |\sigma| + 2$, then

$$\begin{aligned}
\int_1^\infty |(u^{-s/2-1/2} + u^{s/2-1})\psi(u)|\, du &\leq \int_1^\infty (u^{-(\sigma-1)/2-1} + u^{\sigma/2-1})\psi(u)\, du \\
&\leq 2\int_1^\infty u^{k-1} \sum_{n=1}^\infty e^{-\pi n^2 u}\, du \\
&\leq 2\sum_{n=1}^\infty \int_0^\infty u^{k-1} e^{-\pi n^2 u}\, du \\
&= 2\sum_{n=1}^\infty \frac{1}{(\pi n^2)^k} \int_0^\infty u^{k-1} e^{-u}\, du \\
&\leq C\Gamma(k) = C(k-1)! \\
&\leq Ce^{(k-1)\log(k-1)} \leq Ce^{(|\sigma|+1)\log(|\sigma|+1)}.
\end{aligned}$$

This shows that growth defined by the integral is ≤ 1. On the other hand, we want to show that $\rho_{\tilde{\xi}} \geq 1$: using the defining equation for ξ, we have

$$\tilde{\xi}(s) = s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Consider s along the positive real axis, more specifically take $s = 2m$ for $m \in \mathbb{N}$. Note that[1] $\zeta(2m) \to 1$ as $m \to \infty$. So if $|\pi^{-s/2}\Gamma(s/2)| \leq Ae^{B|s|^\rho}$, we have

$$\frac{(m-1)!}{\pi^m e^{2^\rho B m^\rho}} \leq A$$

for all m. Taking $m \to \infty$ shows that $\rho > 1$ (e.g. by ratio test). This concludes that the growth order of $\tilde{\xi}$ is exactly 1.

Next, observe that $\tilde{\xi}$ satisfies the following properties:

[1] e.g. Using Riemann sums, one has $1 + \int_2^\infty t^{-p}\, dt \leq \sum_{n=1}^\infty n^{-p} \leq 1 + \int_1^\infty t^{-p}\, dt$ for $p > 1$.

- $\tilde{\xi}$ is an entire function with zeros precisely the zeros of $\zeta(s)$ in the critical strip: this follows directly from the defining equation of ξ. (So it suffices to show that the zeros of $\tilde{\xi}$ is infinite.)
- $\tilde{\xi}(s) = s(1-s)\xi(s)$ satisfies $\tilde{\xi}(s) = \tilde{\xi}(1-s)$.

Consider the function $F(s) = \tilde{\xi}(s + 1/2)$. By the above, this an even entire function. Define $G(s) = F(s^{1/2})$, which is also entire by an argument as in Tutorial 2 (Problem 3, Step 2) because F is even. Since F has order 1, G has order $1/2$. The following lemma shows that G (and so F and $\tilde{\xi}$) have infinitely many zeros, and thus completes the proof.

Lemma (cf. Chap 5, Ex 14)**.** *If h is entire and of growth order ρ that is non-integral, then h has infinitely many zeros.*

Indeed, if h has finitely many zeros, Hadamard's theorem implies that it can be written as $h(z) = p(z)e^{q(z)}$. But the RHS has growth order $\deg q$ (Ex!), so a contradiction to the assumption ρ is non-integral.